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# Annotating mathematical material: a route to developing holistic understanding and learner autonomy

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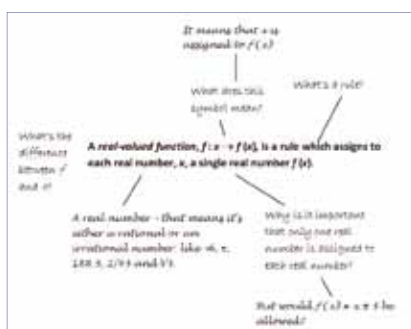


## Introduction

Many students tend to see mathematics as a disparate collection of rules, procedures, theorems, definitions, formulae or applications. Such students believe that mathematical ideas should be memorised, and then used to solve problems. Clearly, though, such a disjointed approach to learning mathematics becomes increasingly untenable as the level of complexity and abstraction increases. But are these students ever likely to appreciate how mathematics fits together as an abstract system of thought if they merely attend lectures and work away on solving problems? It remains difficult to shift ingrained perceptions of our discipline, and it seems unreasonable to expect that it will simply be sufficient to present students with the finished products of mathematical rigour.

Study after study makes it clear [1-3] that we cannot expect an increasingly diverse body of students to make the transition to university-level study without further support. One approach is to help students pick up the strategies that expert mathematicians employ in making sense of a proof or coping with further abstraction (see for instance [4,5, 6]). This can provide a basis for students to approach the varied mathematical tasks they face in a more consistent or coherent fashion. When looking to shift deep-seated attitudes towards mathematics, we need more than an occasional reminder or an introductory course. If one begins to realise ways in which the same logical principles or approaches to exemplifying mathematical concepts apply to a wide range of contexts, then one is a good way to appreciating how mathematics fits together as a discipline. The challenge is to help students engage at a substantive mathematical level.

Fig 1 - Exemplar annotation on a definition from two students



## Student annotations

This article presents one particular approach to this challenge. The idea itself is deceptively simple. Students annotate mathematics. One can require students to annotate their lecture notes with comments, questions, additional steps or reflections on their learning. Solutions to problems and important theorems can be annotated. You can add notes to a mathematical model about the assumptions that underpin the modelling or annotate a definition with a range of examples that satisfy it. The range of material that can be annotated is clearly vast, as are the types of annotations; as figure 1 illustrates.

One can also go about tasks that involve annotating mathematics in a range of ways. A group of students, or even the entire cohort, could annotate the same

material, whether online through a wiki or simply working together on the same piece of paper, thus promoting an approach to peer learning that accommodates different levels of mathematical understanding. The course could include a requirement that students simply complete a certain number of annotations, or one could grade more directly the quality of the annotations made by the students).

There are a number of advantages that stem from this approach. It is a challenge for students to make sense of complex mathematical material within the time-frame represented by a lecture. Our strategy offers a practical means for students to develop their understanding of mathematical material, helping them to work with the material over a longer time frame. It is now increasingly recognised that it is important for students to develop habits of persistence in studying mathematics. But this approach also puts the students centre stage, as they are the ones who need to be creative in proposing annotations. And it will also be important to find consistent ways to annotate mathematics to highlight the underpinning principles, strategies and patterns of thought that give coherence to mathematics.

The existing literature in relation to this idea of annotating mathematics is, however, limited. Fister and McCarthy [7] report on the use within a mathematics classroom of an annotation feature within a Tablet PC that enables the teacher to write on almost any document just as one would annotate printed material. Hwang and Shadiev, meanwhile, report on a study that investigated the effect of pupils in schools annotating mathematics and reviewing each others' annotations [8]. They found significant gains for students who annotated their assignments, and who reviewed these annotations. They also report literature on student annotation in other disciplines. Oscarson [9], for instance, argues that annotations may support the development of language proficiency. They note also that software is being developed that supports student annotation more generally, as with Hylighter [10]. Hwang and Shadiev's study itself involved the use of a web annotation system, VPen.

An associated area, though, that has received wider attention in the mathematics education literature is that of meta-cognition. This term was originally developed by Flavell to refer to knowledge and regulation of one's own cognitive processes [11], with long standing applications within mathematics education [12, 13]. The use of annotation provides an important means to pay explicit attention to such knowledge and regulatory processes. Annotations may help shape student awareness of the strategies and approaches that underpin their mathematical thinking. Hwang and Shadiev further note that perspectives from meta-cognition have informed developments in various subjects relating the use of student annotation [8].

In this article we explore student annotation of mathematics from a systematic perspective. We look for students to do more than make disparate annotations on mathematics, considering rather approaches that will allow for the development of holistic understanding, and that also promote student engagement in a range of ways. In this it will further be important to consider relevant practical and pedagogical issues, as in relation to the use of software, student progression, assessment and resources.

### **Annotation systems**

We refer to a systematic approach to student annotation as an 'annotation system'. In this it is particularly helpful to explore annotation systems that are linked to existing resources or theoretical frameworks, ideally those that have been found to assist student learning. The link to existing materials aimed at students is important in that students are thus provided with assistance in making the annotations. This article thus considers in most detail an annotation system based on a book for students on how to study mathematics and its applications [6]. But before we consider this annotation system, it is helpful to consider some alternatives. One might even consider asking students to develop their own systems, or to propose adaptations to annotation systems already in use.

### **A framework for problem solving**

George Pólya developed perhaps the best known framework to make sense of mathematical problem solving [4]. He divided problem solving into four stages: understanding the problem, devising a plan, carrying out the plan and reviewing one's work. One could ask students to annotate an attempt to solve a problem, with annotations referring to how their work relates to each of Pólya's stages. But this framework relates most directly to problem-solving at large than to mathematics more specifically, and student attempts at problem-solving may follow these stages to a greater or lesser extent. The challenge in part is to find systems that expose key patterns of mathematical thinking.

### **Forms of attention**

We can contrast this relatively limited annotation system with one based around ideas developed by John Mason [14]. He picks out five forms of attention, suggesting that the quality of attention makes a significant difference in helping students to move from an informal understanding of a concept to a more formal one. His concern is to help students pay attention in a similar way to an expert mathematician, through holding wholes, discerning details, recognising relationships, perceiving properties and reasoning on the basis of agreed properties. Mason, for instance, points out that a lecturer may see a mathematical expression as an instance of something more general, while a student may simply see the specific expression itself [15]. A key underpinning form of attention here is to see

connections between different mathematical ideas. One could ask students to make annotations that refer to entire wholes and that expose details within a whole, or that refer to identify relationships with pertinent other mathematical ideas and so on. These different forms of attention thus provide a clear basis on which to annotate mathematical material, assuming that students are able to appreciate the basis for the annotation system.

### The skill-task annotation system

There are many similarities here with the approach taken within the book *Studying Mathematics and its Applications* [6]. This text outlines a series of thought processes that mathematicians commonly employ when seeking to understand or develop an area of mathematics:

- generate examples of concepts;
- connect visual images to associated symbolic representations;
- attend to the precise meaning of symbols;
- identify which basic concepts contribute to more advanced concepts;
- make sense of the logical ideas involved; and,
- make connections between mathematical ideas.

For instance, when seeking to understand a concept, it is often helpful to generate a range of examples of the idea. Or if you have failed to understand a basic concept it will be almost impossible to make sense of a more advanced concept that builds on it. In order to understand the concept of 'addition' – how to add one number to another – you need first to understand the idea of a 'number'. But students cannot rely on the tutor always identifying the problem for them or generating the range of examples. They need to be able to take a look at an advanced concept and identify the contributory concepts, so that they can then make sure they understand these more basic concepts.

These thought processes, which we can also refer to as intellectual skills, are, though, further embedded within higher-level mathematical tasks, namely constructing or understanding a proof, modelling the real world with mathematics or problem solving. In order to solve a problem, engage in mathematical modelling or construct a proof one often needs to gain further mastery of advanced ideas or make connections between mathematical ideas. While one could offer a course on such aspects of mathematical thinking, it is the capacity to employ these strategies across the whole of one's study of mathematics that is particularly significant. It may thus be helpful to integrate their use into student work by requiring students to make annotations that expose these fundamental mathematical thought processes. And similarly there remains scope also to draw student attention through annotations to characteristic features of these wider mathematical tasks, as with Pólya's framework,

a mathematical modelling cycle (e.g. formulate the investigation, introduce concepts, introduce mathematical ideas, determine relationships, solve, interpret and validate) or the given form of a proof.

We suggest that requiring students to make annotations in relation to such skills and to the underlying frameworks for higher-level mathematical tasks provides a helpful basis for developing a more holistic approach to student learning. We can call this the 'skill-task annotation system', given its focus on underlying thought processes and wider mathematical tasks. The box outlines two sets of resources associated with this particular annotation system.

### Resources to support the Skill-Task Annotation System

The study skills text *Studying Mathematics and its Applications* [6] includes chapters on: Using examples, Thinking visually, Coping with symbols, Taking ideas apart, Thinking logically and Making connections, as well as those on Solving problems, Applying mathematics and Constructing proofs. Some material is already available online: <http://www.palgrave.com/skills4study/subjectareas/maths/index.asp>.

Sigma, the Centre for Excellence in Mathematics & Statistics Support, at Coventry University and Loughborough University, has produced a range of relevant leaflets, including those on interpreting diagrams, analysing ideas, creating examples, writing mathematical text, proof, solving problems and the use of symbols. See the maths study skills leaflets at: <http://www.sigma-cetl.ac.uk/>.

### Issues in learning and teaching

It is, however, one thing to offer an annotation system. It is quite another to implement the system on a given programme of study. It will be helpful then to conclude this article by briefly addressing issues that relate to assessment, peer learning and software.

We begin with the immediate problem that it may be difficult to grade student annotations, given that this is an unfamiliar strategy to both staff and students. But it is not essential to develop formal grading criteria. An alternative is to require students to complete annotations to a minimum standard, given that the purpose is to foster learning that can then be assessed in a more standard fashion. Coursework requirements, as Gibbs outlines [16] are an effective way to ensure that students spend appropriate time on the desired tasks, with submission of further assessed work conditional on completion of the requirement. Scope is also present here for sampling of student work, for peer assessment or for Graduate Teaching Assistants to check that the minimum standard has been achieved. This minimum might simply be a given word count against specific elements of the annotation system. One could, though, build in elements of progression as a course unfolds, so that annotations in early courses are a

course requirement, while annotations in later courses are required at a more sophisticated level, and graded.

The manner in which the annotations are completed is also essential to consider, as one may wish students to annotate the same text or share each other's annotations. There are potential advantages from students working together in this way, as when they share the work involved. It will thus be important to consider ways in which software might support student annotation. Hwang and Shadiev [8] review a number of web based systems that allow annotation of student work, while also analysing the use of their own VPen system, which allows students to annotate objects on the web including text and images, making use of highlighting, underlining, allowing annotations to include texts, tables and other features, and private, group or public access. But other options are possible, including the use of features on standard software. Text annotation and 2-D plot annotation became available on Maple 11, with Maple 13 offering 3-D Plot annotations and WYSIWYG document processing features for mathematical text. Integration may also be possible with virtual learning environments: Maple already includes and add-on for use with the Blackboard virtual learning environment.

## Conclusions

We need to find ways forward to help direct the attention of students to essential features of the mathematical ideas they are seeking to master. Marton and Pang [17] argue that guiding the attention of students towards the critical characteristics of the ideas to be mastered constitutes an essential feature of teaching. In seeking to enable students appreciate how mathematics fits together as an abstract system of thought, it is essential that we develop practical ways forward that take us beyond logical presentation of mathematical ideas; as such presentation does not in itself focus attention on the connectivity that underpin mathematics. If the discipline of mathematics is to respond to its ongoing challenges, then such creative approaches are important in going forward.

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